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Multipartite entanglement dynamics and decoherence

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Abstract. We study the dynamics of multipartite entanglement under decoherence induced by the environment consisting of a fermionic bath. Based on the algebraic measure of entanglement-negativity, we analyze the time evolution of entanglement of both pure states and mixed ones, and find that entanglement evolution depends on both bath temperature and the number of qubits of the system. A linear space S_{LDF} which is dynamically decoupled from the environment is identified in the sense of linear entropy to symbolize the environment effect.

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1 Introduction

Entanglement is one key notion that distinguishes the quantum and classical world, and it also plays an important role in quantum information processing (QIP). A lot of work has been carried out on entanglement both theoretically and experimentally [1–5] and it is a promising resource for future applications. One basic problem is how to measure entanglement. We have a clear definition of measure of entanglement based on so-called concurrence [6] for a 2×2 system, while for multipartite systems, the general agreement [7] is that there is a lack of effective techniques for calculating the optimization for mixed states, even though they possess a physical interpretation. Recently, based on the Peres' separability criterion [8], Vidal et al. introduced negativity [9] as a measure of entanglement, which is the equivalent of concurrence when measuring the entanglement of a 2×2 system, and by which one can effectively compute the entanglement of both pure and mixed states of an arbitrary bipartite system with high dimensions.

The dynamics of entanglement is another interesting problem and attracts considerable attention from physicists for understanding the effect of environment on entanglement. Generally speaking, entanglement dynamics describes the behavior and features of the time evolution of the measure of entanglement. Various models [10–13] have been employed to investigate entanglement dynamics based on so-called concurrence limited to a 2×2 system. The multipartite system has not been modelled so far, however, the multipartite system should receive more attention due to the important role it plays in the applications of quantum computation. One of most recent works concerning multipartite entanglement dynamics was developed in [7], whose authors considered multipartite entanglement dynamics based on generalized concurrence limited to pure quantum states.

In the present paper, we first derive the time evolution of the natural basis of a matrix in the system space under the decoherence induced by the environment, consisting of a fermionic bath. Then by using the negativity [9], we can give the multipartite entanglement dynamics of states either pure or mixed in principle. In particular, the entanglement dynamics of the GHZ state and Werner state [14] are discussed in detail. A linear space S_{LDF} which is dynamically decoupled from the environment has been identified when we use linear entropy to symbolize the environment effect. Under a stronger condition, the state in S_{LDF} evolves unitarily and is therefore decoherence free. The W state and the diagonal states are just such some examples. In principle, making use of our method, one can research the entanglement dynamics and coherence evolution for an arbitrary state in the multipartite systems, either fermionic or bosonic under the decoherence induced by the environment consisting of a fermionic bath.

2 Hamiltonian evolution

Here, we consider the model as a generalization of the work [15] from a 2-qubit system to a multipartite system while keeping the environment unchanged. The Hamiltonian of our model is $H = H_s + H_{sB} + H_B$, where H_s, H_{sB} and H_B denote the Hamiltonian of the system, system-bath interaction and bath respectively, and they read

$$H_s = -\sum_{i=1<j}^{n} \xi_{ij} S_i^z S_j^z \tag{1a}$$

$$H_{sB} = -\frac{J_0}{\sqrt{N}} \left(\sum_{i=1}^{n} S_i^z \right) \sum_k S_{Bk}^z \tag{1b}$$

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$$H_B = -w\sum_k S^x_{Bk} - \frac{J}{N}\sum_{k,k'} S^z_{Bk}S^z_{Bk'} \tag{1c}$$

where ξ_{ij} are the coupling constants between qubit i and qubit j, n is the number of qubits in the system we consider to be investigated. J_0, J are exchange coupling constants and ω is the strength of the transverse field. All of them are non-negative constants. The indices of the sums k, k' run from 1 to N.

As is well-known, the state ρ_s for a n-qubit system can always be expanded by using a natural basis $E = \{E_{\alpha\beta}\}$, where $E_{\alpha\beta} = |\alpha\rangle\langle\beta| = |\alpha_1\alpha_2...\alpha_n\rangle\langle\beta_1\beta_2...\beta_n|$ with every α_i and β_j taking values of 0 or 1, and the bit-string $\alpha_1\alpha_2...\alpha_n$, $\beta_1\beta_2...\beta_n$ taken over all the possible permutations of 0 and 1

$$\rho_s = \sum_{\alpha,\beta} c_{\alpha\beta}E_{\alpha\beta}. \tag{2}$$

Assuming the bath density matrix to be a thermal distribution $\rho_B = (\mathrm{e}^{-H_B/T})/Z$, with T the temperature multiplied by the Boltzman constant and $Z = \mathrm{Tr}(\mathrm{e}^{-H_B/T})$, we can write the general density matrix under this bath as $\rho(0) = \rho_s \otimes \rho_B$, where 0 denotes the initial time. Note that we are only interested in the system density matrix evolution $\rho_s(t)$, which can be obtained by tracing out the degree of freedom of the environment. Thus, $\rho_s(t)$ can come down to the evolution of $E_{\alpha\beta}$ with time, i.e.

$$\rho_s(t) = \sum_{\alpha,\beta} c_{\alpha\beta}E_{\alpha\beta}(t) = \sum_{\alpha,\beta} c_{\alpha\beta}\mathrm{Tr}_B\{\mathrm{e}^{-\mathrm{i}Ht}E_{\alpha\beta}\otimes\rho_B\mathrm{e}^{\mathrm{i}Ht}\} \tag{3}$$

where $c_{\alpha\beta}$ is a constant. Obviously,

$$\begin{aligned}
E_{\alpha\beta}(t) =& \frac{1}{Z}\mathrm{Tr}_B\{\mathrm{e}^{-\mathrm{i}(H_{sB}+H_B^{mf})t}|\alpha_1\alpha_2...\alpha_n\rangle\langle\beta_1\beta_2...\beta_n| \\
&\times \mathrm{e}^{-H_B^{mf}/T}\mathrm{e}^{\mathrm{i}(H_{sB}+H_B^{mf})t}\}\mathrm{e}^{\frac{\mathrm{i}t}{2}\sum_{i=1<j}^{n}\xi_{ij}[(-1)^{(\alpha_i+\alpha_j)}-(-1)^{(\beta_i+\beta_j)}]} \\
=& \frac{\mathrm{e}^{-m^2JN/T}}{Z}\mathrm{Tr}_B\Bigg[\exp\Bigg\{\mathrm{i}t\sum_k\Bigg[\Bigg(\sum_i^n\frac{J_0}{\sqrt{N}}S_i^z + 2mJ\Bigg)S^z_{Bk} \\
&+ \omega S^x_{Bk}\Bigg]\Bigg\}|\alpha_1\alpha_2...\alpha_n\rangle\langle\beta_1\beta_2...\beta_n| \\
&\times \exp\Bigg\{\sum_k(\omega S^x_{Bk} + 2mJS^z_{Bk})/T\Bigg\} \\
&\times \exp\Bigg\{-\mathrm{i}t\sum_k\Bigg[\Bigg(\sum_i^n\frac{J_0}{\sqrt{N}}S_i^z + 2mJ\Bigg)S^z_{Bk} + \omega S^x_{Bk}\Bigg]\Bigg\}\Bigg] \\
&\times \exp\Bigg\{\frac{\mathrm{i}t}{2}\sum_{i=1<j}^{n}\xi_{ij}\Big[(-1)^{(\alpha_i+\alpha_j)} - (-1)^{(\beta_i+\beta_j)}\Big]\Bigg\} \\
=& E_{\alpha\beta}\Bigg[2\cosh\Bigg(\frac{\Theta}{2T}\Bigg)\Bigg]^{-N}\prod_k\mathrm{Tr}_B\{\mathrm{e}^{\mathrm{i}I'}\mathrm{e}^{R}\mathrm{e}^{\mathrm{i}\widetilde{I}}\} \\
&\times \exp\Bigg\{\frac{\mathrm{i}t}{2}\sum_{i=1<j}^{n}\xi_{ij}\Big[(-1)^{(\alpha_i+\alpha_j)} - (-1)^{(\beta_i+\beta_j)}\Big]\Bigg\}
\end{aligned} \tag{4}$$

where

$$H_B^{mf} = -\omega\sum_k S^x_{Bk} - 2Jm\sum_k S^z_{Bk} + m^2JN \tag{5}$$

which is the surrogate of H_B under the mean field approximation (MF) [16], in order to eliminate the nonlinear term in H_B to obtain an analytical result for $\rho_s(t)$. In equation (5) m is the order parameter of the phase transition. The absolute value of m ranges from 0 to 1/2 as long as the temperature ranges from the critical value $T_c = J/2$ to 0. We obtain a Curie-Weiss equation in [16] by using MF

$$\frac{J}{\Theta} = \tanh\frac{\Theta}{2T} \tag{6}$$

where $\Theta = \pm\sqrt{\omega^2 + 4m^2J^2}$. If we make the substitution $m \to -m$ or $\Theta \to -\Theta$, everything will remain for H_B z symmetry, so we take positive values of them for convenience below. And thus

$$\begin{aligned}
I' &= t\Bigg[\Bigg(\frac{(n-2n_\alpha)J_0}{2\sqrt{N}} + 2mJ\Bigg)S^z_{Bk} + \omega S^x_{Bk}\Bigg] \\
R &= (\omega S^x_{Bk} + 2mJS^z_{Bk})/T \\
\widetilde{I} &= -t\Bigg[\Bigg(\frac{(n-2n_\beta)J_0}{2\sqrt{N}} + 2mJ\Bigg)S^z_{Bk} + \omega S^x_{Bk}\Bigg].
\end{aligned} \tag{7}$$

In equation (7), we have defined

$$n_\alpha = \alpha_1 + \alpha_2 + ... + \alpha_n, \quad n_\beta = \beta_1 + \beta_2 + ... + \beta_n. \tag{8}$$

Thus we get the $E_{\alpha\beta}(t)$ with the following expression for N being large

$$\begin{aligned}
E_{\alpha\beta}(t) =& E_{\alpha\beta}A_{n_\beta-n_\alpha}(t)\exp\Bigg\{\frac{\mathrm{i}t}{2}\sum_{i=1<j}^{n}\xi_{ij}\Big[(-1)^{(\alpha_i+\alpha_j)} \\
&- (-1)^{(\beta_i+\beta_j)}\Big]\Bigg\} = E_{\alpha,\beta}A_{n_\beta-n_\alpha}(t)\mathrm{e}^{\mathrm{i}\phi t}
\end{aligned} \tag{9}$$

where

$$\begin{aligned}
A_{n_\beta-n_\alpha}(t) = \Bigg[&\cos\Bigg(\frac{(n_\beta-n_\alpha)tmJJ_0}{\Theta\sqrt{N}}\Bigg) \\
&+ \mathrm{i}\frac{\Theta}{J}\sin\Bigg(\frac{(n_\beta-n_\alpha)tmJJ_0}{\Theta\sqrt{N}}\Bigg)\Bigg]^N
\end{aligned} \tag{10}$$

$\phi = \frac{1}{2}\sum_{i=1<j}^{n}\xi_{ij}[(-1)^{(\alpha_i+\alpha_j)} - (-1)^{(\beta_i+\beta_j)}]$
and the module of $A_{n_\beta-n_\alpha}(t)$

$$|A_{n_\beta-n_\alpha}| \approx \exp\Bigg(-\frac{(n_\beta-n_\alpha)^2t^2J_0^2m^2}{2}\Bigg(\frac{J^2}{\Theta^2}-1\Bigg)\Bigg). \tag{11}$$

Obviously, the time evolution of the density matrix can be divided into two terms including a decay term $A_{n_\beta-n_\alpha}$ induced by the environment and the term of phase factor ϕ generated by the coupling between qubits.

3 Entanglement dynamics

As above, we have obtained the final time-dependent reduced system density matrix $\rho_s(t)$ via which the dynamics of entanglement under decoherence can be understood clearly. Here, we use the negativity [9] as a measure of entanglement for its ease of computation. Given the system density matrix ρ_s, the entanglement of ρ_s can be defined as the negativity

$$N(\rho_s) = \frac{\|\rho_s^{T_i}\| - 1}{2} \quad (12)$$

which corresponds to the absolute value of the sum of negative eigenvalues of $\rho_s^{T_i}$ [17], and which vanishes for unentangled states. In equation (12), $\|\rho_s^{T_i}\|$ means the sum of absolute values of eigenvalues of $\rho_s^{T_i}$, and $\rho_s^{T_i}$ which is the partial transpose of ρ_s with respect to part i. Negativity is a measure of entanglement that can be computed effectively for any pure or mixed state of an arbitrary bipartite system, and does not increase under LOCC (local operation amended by classical communication) [9].

Negativity is used to quantify the degree to which $\rho_s^{T_i}$ fails to be positive and represents the strength of quantum correlation between part i and the sum of other components of the system. According to [18,19], we can classify the entanglement properties of a multipartite state by considering the different bipartitions of the system, so we can use $N_{m:n-m}$ to measure the strength of quantum correlation between one group with m particles and the other groups with n-m particles. Similarly, we can incorporate this into the reduced density matrix obtained by tracing over some subsystems. For example, in the case of a tripartite system with density matrix ρ_{ABC}, there will be 6 splittings namely *AB-C, BC-A, AC-B*, and *A-B, B-C, A-C* after tracing one subsystem. Any splitting of a multipartite system will have a negativity, so the number of negativities equals that of splittings. So we can calculate all the negativities to quantify the entanglement of any state.

In practice, we are interested in some explicit examples. Here, we firstly consider the entanglement dynamics of two types of initial pure states: the GHZ state and, the W state, — which are known to bear incompatible multipartite correlations, in the sense that they can not be transformed into each other by local operations and the classical communication [19]. Secondly, we discuss the entanglement evolution of Werner state as an example of a mixed state under the environment.

Case 1. Let the initial system state be a n-qubit GHZ state $|\psi_n\rangle_{GHZ} = (|00...0\rangle + |11...1\rangle)/\sqrt{2}$. We label the negativity as $N_{1:n-1}$ to quantify the quantum correlation between a single party and the remainder of $n-1$ parties. With equation (12), the negativities can be calculated easily for the highly symmetric structure of GHZ state and we find that all the negativities between any party and the remainder are the same with expression

$$N_{1:n-1} = \frac{1}{2}\exp\left[-\frac{(2n)^2 t^2 J_0^2 m^2}{8}\left(\frac{J^2}{\Theta^2} - 1\right)\right]. \quad (13)$$

Obviously, the entanglement shows an exponential decay due to the system-bath interaction. Similarly, we can obtain all the negativities $N_{2:n-2}$ between a subsystem with 2 parties and the other subsystem with $n-2$ parties and find that all the negativities $N_{2:n-2}$ take the same value with $N_{1:n-1}$. Furthermore all the negativities of any bipartition of n parties have the same value as that shown in equation (13), i.e,

$$N_{m:n-m} = N_{1:n-1} = \frac{1}{2}\exp\left[-\frac{(2n)^2 t^2 J_0^2 m^2}{8}\left(\frac{J^2}{\Theta^2} - 1\right)\right] \quad (14)$$

where m is an integer which ranges from 1 to $n/2$ when n is even, and 1 to $(n-1)/2$ when n is odd. Such a result is of no surprise, because we can always consider all the bipartitions of state ρ_s as the simplest example $|B\rangle = (|u_a u_b\rangle + |d_a d_b\rangle)/\sqrt{2}$, where the sub-indices a, b represent one group with m parties and the other group with n-m parties. Now we should also consider the entanglement properties of a reduced density matrix ρ_r obtained by partial tracing over some subsystems. Because of the fact that the partial tracing of some subsystems from the system state ρ_{GHZ} will make the reduced matrix $\rho_r(t)$ possess no quantum correlation, the entanglement of $\rho_r(t)$ is 0. With the above analysis, for convenience, we take $N(t)$ which equals to $N_{m:n-m}$ as the entanglement evolution of the GHZ state and plot it in Figure 1.

As a first observation illustrated in the sub-figure of Figure 1, the evolution of entanglement depends on temperature; the lower the temperature of the bath is, the longer the entanglement remains. We also find that only in the limit $t \to \infty$, will the entanglement vanish, that is $\rho_{GHZ}(t \to \infty) = (|00,...,00\rangle\langle 00,...,00| + |11,...,11\rangle\langle 11,...,11|)/2$ which is a completely separable state. In the sub-figure of Figure 1, we find that the larger the number of qubits a system consists of, the weaker the negativity of system is under the environment.

Case 2. Compared to the entanglement of GHZ state, the entanglement of a n-qubit W state is more complicated. We can make some discussion about the influence on W state imposed by the environment. When the initial system state is a n-qubit W state,

$$|\psi_n\rangle_W = (|00...01\rangle + |00...10\rangle + ... + |10...00\rangle)/\sqrt{n}. \quad (15)$$

Thus the density matrix of W state has the expression $\rho_W(0) = \sum_{\alpha,\beta}(1/n)E_{\alpha,\beta}$, and the time-dependent reduced density matrix can be expressed as

$$\rho_W(t) = \sum_{\alpha,\beta}\frac{1}{n}E_{\alpha,\beta}A_{n_\beta - n_\alpha}(t)\mathrm{e}^{\mathrm{i}\phi t} \quad (16)$$

where $\phi = (1/2)\sum_{i=1<j}^{n}\xi_{ij}[(-1)^{(\alpha_i+\alpha_j)} - (-1)^{(\beta_i+\beta_j)}]$. Obviously, $n_\alpha = n_\beta$ for W state, so $A_{n_\beta - n_\alpha}(t) = A_0(t) = 1$, and

$$\rho_W(t) = \sum_{\alpha,\beta}\frac{1}{n}E_{\alpha,\beta}\mathrm{e}^{\mathrm{i}\phi t}. \quad (17)$$

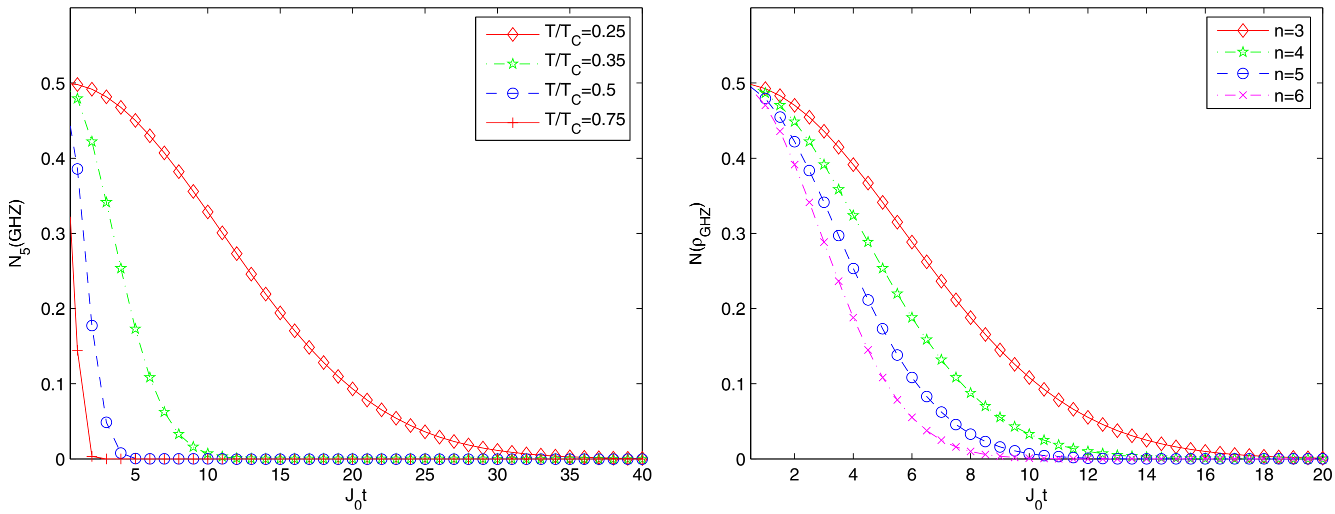

Fig. 1. $N(t)$ versus the scaled time J_0t. The left sub-figure illustrates the entanglement evolution of given GHZ states with five qubits under different temperatures and the right one describes the evolution of entanglement of GHZ state with different number of qubits under a certain temperature $T/T_C = 0.35$ respectively. $J = 2, \omega = 0.1$.

It is easy to find that the environment does not affect the system state, but the coupling between qubits induces the phase factor ϕ in the reduced density matrix. Note the fact that the calculation of entanglement of W state of multipartite system is a complicated task; here, we just make some discussion about the properties of entanglement evolution of the W state. Since the fact that $A_{n_\beta - n_\alpha} = 1$ and the phase factor $\phi \neq 0$, the entanglement will not vanish but oscillate or remain constant. If all of the coupling constants between qubits take the same value ξ_0, then the phase factor ϕ will vanish and the W state will remain unchanged, so entanglement will be unchanged for all time.

Case 3. We will pick out one mixed state, i.e., a n-qubit Werner state

$$\rho_{Werner} = \frac{(1-p)\ I}{2^n} + p|\phi_\pm\rangle\langle\phi_\pm|$$

where p is a real parameter which ranges from 0 to 1, I is an identity matrix with 2^n dimensions, and $|\phi_\pm\rangle = (|00...0\rangle \pm |11...1\rangle)/\sqrt{2}$. Through a similar calculation, we get the entanglement time evolution of any bipartition between a single party and the remainder of $n-1$ parties with the same expression as

$$N_{1:n-1} = \frac{p}{2}\exp\left[-\frac{(2n)^2t^2J_0^2m^2}{8}\left(\frac{J^2}{\Theta^2}-1\right)\right] - \frac{1-p}{2^n}. \tag{18}$$

Since negativity is a positive value, equation (18) holds only under the condition

$$\frac{p}{2}\exp\left[-\frac{(2n)^2t^2J_0^2m^2}{8}\left(\frac{J^2}{\Theta^2}-1\right)\right] > \frac{1-p}{2^n}$$

otherwise, the entanglement is 0. Similar to that of GHZ state, the entanglement evolution of a Werner state with any bipartition m:n-m takes the same value as $N_{1:n-1}$, while, any reduced matrix of a Werner state after tracing one or more subsystems is a completely separable density matrix. Therefore we take $N(t)$ to be the same expression of $N_{1:n-1}$ in equation (18) as the entanglement evolution of a Werner state. The entanglement dynamics of the Werner state is illustrated in Figure 2 where we can see the entanglement evolution behavior under changing environment. Obviously, the entanglement of a Werner state evolves differently from that of GHZ state, with regards to the Werner state, a finite time is needed to corrupt the entanglement; while for the GHZ state, the time taken for entanglement destruction is infinite. In this sense, we can say the Werner state is weaker than the GHZ state in our model.

In principle, we can give the entanglement evolution of any n-qubit system state under the environment, taking the complexity of the calculation into account. Here we omit other examples.

4 Decoherence-free linear space

The results of case (2) in Section 3 imply that some states with special symmetry such as the W state will not be affected by the environment. Finding out the class of states which do not perceive the environment is an interesting problem.

Generally, the mixed states as well as pure states will decohere when exposed to the environment. Here, we consider the general case of either a pure or a mixed system state exposed to the environment. As we know, usually pure states will become mixed when interacting with the environment, and the mixed states become more mixed than ever for the system-bath interaction; thus, we can use the evolution of purity of a state to estimate the decoherence under the environment. A proper candidate used to measure the effect of the environment on the system is linear entropy [20]. Linear entropy has been applied to measure both entanglement and decoherence for different

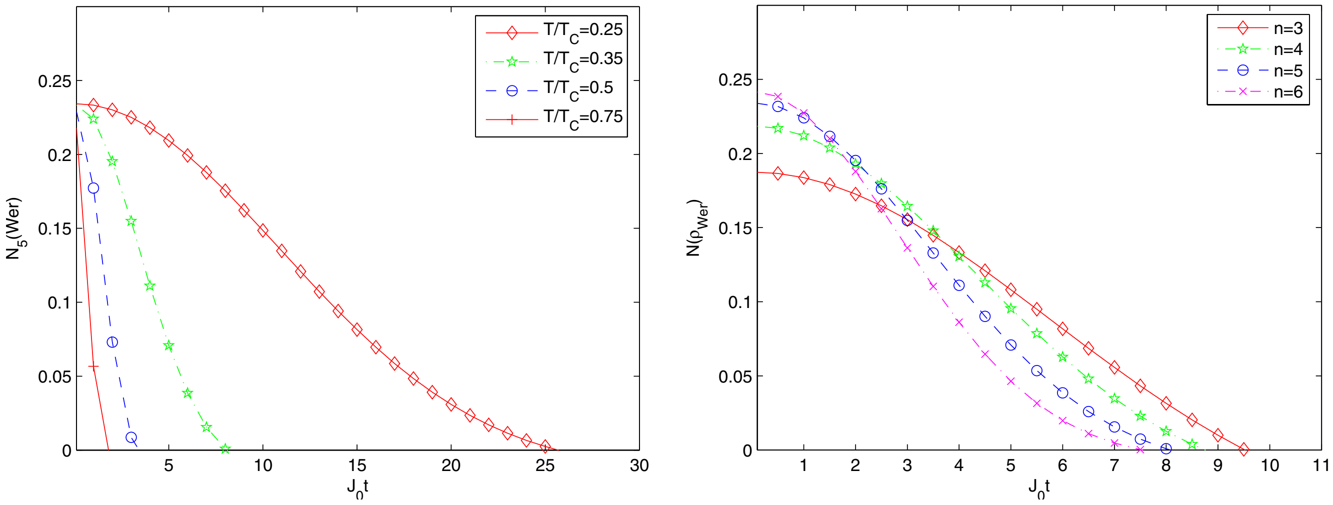

Fig. 2. $N(t)$ of Werner state versus the scaled time J_0t. The left sub-figure describes the entanglement time evolution of a Werner state with five qubits under different bath temperatures, and the time evolution of entanglement of Werner state with different qubits under a fixed bath temperature $T/T_C = 0.35$ is illustrated on the right sub-figure. $p = 1/2, J = 2, \omega = 0.1$.

perspectives. In view of the specific form of linear entropy, it is a function of purity of state and the purity is a reliable expression to represent the effect of the environment; so linear entropy is often used to measure decoherence. A similar study has been carried out on decoherence by using linear entropy in [21,22]. In this paper, we will employ linear entropy to estimate the decoherence of the multi-qubit system in a fermionic environment. From the results above, we find the decoherence-free linear space defined as S_{LDF} of our model.

Lemma. S_{LDF} is a linear space such that every element in it can be expanded by the set of basis $E_{DF} = \{E_{\zeta\eta}\}$ and $n_\zeta = n_\eta$, where $E_{\zeta\eta} = |\zeta\rangle\langle\eta| = |\zeta_1\zeta_2...\zeta_n\rangle\langle\eta_1\eta_2...\eta_n|$ with every ζ_i and η_j taking values of 0 or 1, the bit-string $\zeta_1\zeta_2...\zeta_n$, $\eta_1\eta_2...\eta_n$ taken over all the possible permutations of 0 and 1, and $n_\zeta = \zeta_1 + \zeta_2 + ... + \zeta_n$, $n_\eta = \eta_1 + \eta_2 + ... + \zeta_n$.

Proof. Since $n_\zeta = n_\eta$, we get $A_{n_\eta - n_\zeta}(t) = 1$, and thus

$$E_{\zeta\eta}(t) = E_{\zeta\eta} \times \exp\left\{\frac{\mathrm{i}t}{2}\sum_{i=1<j}^{n}\xi_{ij}\left[(-1)^{(\zeta_i+\zeta_j)} - (-1)^{(\eta_i+\eta_j)}\right]\right\} \quad (19)$$

where $\zeta_1, \zeta_2, ..., \zeta_n, \eta_1, \eta_2, ..., \eta_n$ take values 0 or 1, then we can express the time evolution of basis $E_{\zeta\eta}(t)$ as

$$E_{\zeta\eta}(t) = E_{\zeta\eta}\mathrm{e}^{\mathrm{i}\phi t} \quad (20)$$

where $\phi = \sum_{i=1<j}^{n}\xi_{ij}[(-1)^{(\zeta_i+\zeta_j)} - (-1)^{(\eta_i+\eta_j)}]$ is a real phase factor, and thus

$$E_{\eta\zeta}(t) = E_{\eta\zeta}\mathrm{e}^{-\mathrm{i}\phi t}. \quad (21)$$

Obviously, the effect induced by coupling between qubits on a state belonging to S_{LDF} is only the production of a real phase factor, and the amplitude of basis E_{DF} is invariant under the environment. Since any state density matrix can be expanded by the set of basis E_{DF} $\rho_s = \sum_{\zeta,\eta} c_{\zeta\eta}E_{\zeta\eta}$ thus the linear entropy of the time-dependent expression can be given as

$$S(t) = 1 - \mathrm{Tr}(\rho_s^2(t)) = 1 - \sum_{\zeta,\eta}|c_{\zeta\eta}|^2 = S(0). \quad (22)$$

So in the sense of linear entropy used to express the environment effect, we can conclude that the linear space S_{LDF} is decoherence free. The proof is completed.

With the lemma, we can say that a state density matrix which can expanded by the set of basis E_{DF} does not perceive the presence of the environment in the sense of linear entropy to estimate decoherence. In contrast, any other state which cannot be expanded by the set of basis E_{DF} will decohere in general and become a state belonging to S_{LDF} after a long time under the environment, since the module $A_{n_\eta - n_\zeta}$ when $\eta \neq \zeta$ is an exponential decay with time.

As an example, we consider the n-qubit W state $|\psi_n\rangle_W = (|00...01\rangle + |00...10\rangle + ... + |10...00\rangle)/\sqrt{n}$ as the initial system state. Since the W state can be expanded by the set of basis E_{DF}, thus, the linear entropy of W state will be invariant under the decoherence induced by the environment, i.e., $S(\rho_W(t)) = S(\rho_W(0))$. In the sense of linear entropy as a measure of decoherence, we can see that the W state is free from the environment.

Likewise, we can also check that the mixtures of the diagonal states such as the W state do not perceive the presence of the environment because such mixed states are still able to be expanded by the set of basis E_{DF}. Here, the diagonal states mean one class of states whose density matrices are diagonal.

In particular, when all the ξ_{ij} take the same value ξ_0 which is a constant, any state in S_{LDF} will keep invariant under the environment and a series of completely decoherence-free subspaces [23–25] can be found. They are defined as $\mathcal{S}_{n_\alpha}$ constructed by orthogonal quantum states with the same value of n_α, and the dimension of $\mathcal{S}_{n_\alpha}$ is $n!/[n_\alpha!(n-n_\alpha)!]$. For example, $\mathcal{S}_n$ is constructed by one element $|11,...,1\rangle$. Any pure state of the subspace $\mathcal{S}_{n_\alpha}$ can be expanded by the basis of $\mathcal{S}_{n_\alpha}$, therefore, the pure states belonging to $\mathcal{S}_{n_\alpha}$ do not perceive the presence of the environment. Our result as a particular instance is consistent with the well established theory on decoherence free subspaces and it is important for us to deal with error correction. Information is encoded in subspaces $\mathcal{S}_{n_\alpha}$ (codes) of the total Hilbert space in a way that errors induced by the interaction with the bath can be detected and corrected. The important point is that the detection of errors, if they belong to the class of errors correctable by the given code, should be performed without gaining any information about the actual state of the computing system prior to decoherence.

5 Conclusion

Based on the use of negativity as a measure of entanglement, we have studied the entanglement dynamics of multipartite-qubit system with an initial state either pure or mixed in a symmetry-broken environment. Notably, we analyze the entanglement dynamics of GHZ state and Werner state in detail, and find that the time to corrupt the entanglement of them is completely different. For GHZ state, the time is infinite, while the environment can remove the entanglement of a Werner state in a very limited time. Our model resembles some selected atoms interacting with a spin-like bath consisting of a larger number of two-level Rydberg atoms.

Due to the system-bath interaction, and noting that the module of $A_{n_\beta - n_\alpha}$ is an exponential decay, the states which cannot be expanded by the set of basis of E_{DF} will lose coherence in general and become states belonging to the linear space S_{LDF}. Any other state which can be expanded by the set of basis of E_{DF} does not perceive the presence of the environment in the sense of linear entropy to measure decoherence. Thus we are allowed, in principle, to design noiseless quantum codes which are of importance in QIP. The entanglement dynamics not only depends on the bath temperature, but also on the number of qubits of the system.

Our research can be also applied to multipartite systems with high dimensions under a spin-like environment, so this paper will shed some light on the entanglement dynamics and decoherence of multipartite system either fermionic or bosonic with high dimensions. In the end, we think that our analysis will contribute to a better understanding of entanglement dynamics and decoherence of multipartite systems.

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